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SHOCK-WAVE DIFFRACTION BY A WEDGE MOVING AT SUPERSONIC SPEED

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SHOCK-WAVE DIFFRACTION BY A WEDGE MOVING AT SUPERSONIC SPEED

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ABSTRACT: The paper studies the diffraction of a shock wave of arbitrary strength caused by the upper surface of a wedge moving at supersonic speed by assuming that the intensity of the shock wave and the attached compression shock as well as the wedge angle and the angle of shock incidence differ but little from each other.

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SHOCK-WAVE DIFFRACTION BY A WEDGE MOVING AT SUPERSONIC SPEED

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A study is made of the diffraction of a shock wave of arbitrary intensity by the up- /631*
per surface of a wedge moving at supersonic speed, under the assumption that the intensity of the shock wave and of the attached compression shock and also the wedge angle α and the angle of shock incidence δ are but little different from each other (Fig. 1).

The case of flow over a wedge moving at supersonic speed with a plane shock wave incident on it, without diffraction, was examined in [1]. In this paper conditions were obtained for the case of flow with constant parameters in a region AFK bounded by the on-coming shock wave, the compression shock attached to the wedge and the wall of the wedge.

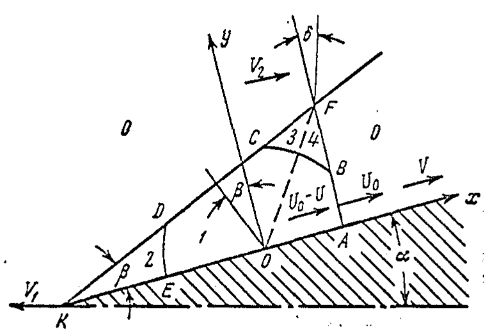

Fig. 1.

Studied in [2] was the diffraction of a shock wave of arbitrary intensity by a thin wedge moving at supersonic speed; in [3] a study was made of the diffraction of a weak wave by a slender wedge moving at hypersonic speed; also studied** was the diffraction of a weak wave by an arbitrary wedge moving at supersonic speed.

1. Statement of the Problem. Obtained as a result of the application of perturbations to a flow with constant parameters is a diffraction pattern bounded by the shock wave AB, the compression shock CD, the wall of the wedge AE and the arcs BC and DE of a Mach circle whose center O moves along the wall of the wedge at a flow speed of U_0-U behind the shock wave AF, where U_0 is the velocity of the shock wave and U is the velocity of the shock wave relative to the flow behind it. The problem is formulated in a coordinate system Ox'y' linked to the moving center of perturbation O, coincides with the point of intersection of the bisector of the angle AFK with the wall of

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*Numbers in the margin indicate pagination in the foreign text.

**S. M. Ter-Minasyants: Diffraction of a plane wave by a wedge moving at supersonic speed. Candidate's dissertation, Moscow State University, 1967.

the wedge. In this coordinate system the unperturbed gas in the region AFK is in a state of rest. The problem being examined is self-similar in time t. We linearize the equations of plane unsteady gas motion and introduce the dimensionless variables

$$u = \frac{u'}{a_1}, \quad v = \frac{v'}{a_1}, \quad p = \frac{p'}{a_1^2 \rho_1}, \quad x = \frac{x'}{a_1 t}, \quad y = \frac{y'}{a_1 t}$$

where u', v', p' are the perturbed velocity components and the pressure perturbation, and a_1 and ρ_1 are the speed of sound and the unperturbed density in region 1. /632

We represent the equation of the perturbed front of the shock wave AF in the form

$$x = k + \psi(y)$$

When x = k the relations on the shock wave AB will take the form

$$u = \frac{2}{\gamma+1} \frac{(M_0 - M)^2 + 1}{(M_0 - M)^2} (\psi(y) - y\psi'(y)) + D_1$$

$$v = (k - m(M_0 - M))\psi'(y) - v_0, \qquad p = \frac{4k}{\gamma+1}(\psi(y) - y\psi'(y)) + E_1 \tag{1.1}$$

$$\left(M_0 = \frac{U_0}{a_0}, \ M = \frac{V}{a_0}, \ k = \frac{U}{a_1}, \ m = \frac{a_0}{a_1}\right)$$

Here a_0 and V are the speed of sound and the flow velocity in the region O, D_1 and E_1 are known constants dependent on the perturbed parameters u_0, v_0, ρ_0 and p_0 ahead of the shock wave in the region O.

Equations (1.1) can be written in the form

$$u = A_1 p + F_1, \quad y\frac{\partial v}{\partial y} = B_1 \frac{\partial p}{\partial y} \quad \text{for } x = k$$

$$A_1 = \frac{(M_0 - M)^2 + 1}{2k(M_0 - M)^2}, \quad B_1 = \frac{\gamma+1}{2} \frac{(M_0 - M)^2 - 1}{(\gamma - 1)(M_0 - M)^2 + 2} \tag{1.2}$$

Here F_1 is a known constant and γ is the heat capacity ratio.

Analogous relations can be written on the compression shock KF for $y \cos\beta - x \sin\beta = k$.

The flow in region 2 is known and corresponds to flow over a wedge at a velocity V_2-V_1, where V_1 is the velocity of the wedge and V_2 is the flow velocity behind the incident shock wave.

Flows with constant parameters in the regions 3 and 4 are completely determined if relations (1.1) are written on the slightly divergent rectilinear segments:

shock wave FB

$$x = k - (y - y_F) \operatorname{tg} \varepsilon_1$$

compression shock FC

$$y \cos\beta - x \sin\beta = k + (y - y_F) \operatorname{ctg} (\beta - \varepsilon_2)$$

the condition on the weak tangential discontinuity FO for $y = x \tan 1/2\ \theta_0$ is

$$u_3 y_F - v_3 x_F = u_4 y_F - v_4 x$$
$$(x_F = k,\ y_F = k \operatorname{tg} {}^1\!/_2\, \theta_0,\ \theta_0 = {}^1\!/_2\, \pi + \beta)$$

on which the pressure and the pressure derivatives are constant [2].

These relations yield seven conditions for determination of the seven unknowns u_3, u_4, v_3, v_4, ε_1, ε_2 and $p_3 = p_4$, where ε_1 and ε_2 are small angles of divergence of the shock wave and the compression shock.

2. Formulation of the Problem for the Function p. After linearization of the equation and Chaplygin transformation the problem reduces to a Laplace equation for the pressure perturbation. The region corresponding to the diffraction region goes over to an or- /633
thogonal curvilinear pentagon ABCDE in the plane $z = r \exp i\,\theta = \mu + i\nu$, bounded by four arcs of a circle and a straight line (Fig. 2).

The boundary conditions are written for the normal and tangential components of the pressure derivatives [4]

$$a \frac{\partial p}{\partial n} + b \frac{\partial p}{\partial s} = 0$$

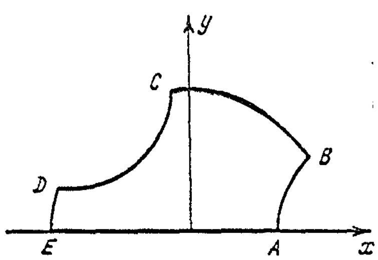

Fig. 2.

Here

$$\begin{array}{lll} a = \vartheta\,(\theta,\ 0), & b = 1 & \text{on } AB \\ a = 0, & b = 1 & \text{on } BC \text{ and } DE \\ a = \vartheta\,(\theta,\ \theta_0), & b = 1 & \text{on } CD \\ a = 1, & b = 0 & \text{on } AE \end{array}$$

where

$$\vartheta\,(\theta,\ 0) = \frac{\sqrt{1 - k^2 \sec^2\theta}}{kA_1 \operatorname{tg}\theta - B_1 \operatorname{ctg}\theta}, \qquad \vartheta\,(\theta,\ \theta_0) = \frac{\sqrt{1 - k^2 \sec^2 (\theta - \theta_0)}}{kA_1 \operatorname{tg} (\theta - \theta_0) - B_1 \operatorname{ctg} (\theta - \theta_0)}$$

and the equations of the arcs AB and CD of the circles have the form, resp.,

$$k\,(1 + r^2) = 2r \cos\theta, \qquad k\,(1 + r^2) = 2r \cos(\theta - \theta_0)$$

The coefficient a of $\partial p/\partial n$ goes to infinity at the points $N \in AB$, L and $Q \in CD$ for

$$\theta_N = \operatorname{arc\,tg} \sqrt{B_1 / kA_1}, \qquad \theta_{L,Q} = \theta_0 \mp \operatorname{arc\,tg} \sqrt{B_1 / kA_1}$$

Integrating the second condition of (1.2) along the shock wave AB and considering that y = k tan x when x = k, we get

$$k^{-1} B_1 \int_{AB} \operatorname{ctg} \theta \, dp = v_B - v_A \tag{2.1}$$

The conditions which must be fulfilled along the compression shock CD have the form

$$\begin{gathered} k^{-1} B_1 \int_{CD} \operatorname{ctg} (\theta - \theta_0) \, dp = (u_C - u_D) \cos \beta + (v_C - v_D) \sin \beta \\ \int_{CD} dp = p_2 - p_3 \end{gathered} \tag{2.2}$$

The boundary problem we have obtained can be solved by mapping the curvilinear pentagon ABCDE onto the upper half-plane.

3. Construction of the Function that Maps the Curvilinear Pentagon onto the upper Half-Plane. By means of the bilinear transformation

$$\zeta = \frac{1 - (k + i k_1) z}{z - k - i k_1}, \qquad k_1 = \sqrt{1 - k^2}$$

the curvilinear pentagon ABCDE is mapped onto the second quadiant minus one quarter of the unit circle with center at the origin and minus one-half of a circle of radius d with center on the real axis at a distance of -c from the origin (Fig. 3) /634

$$d = \frac{k_1}{k + k \sin \beta - k_1 \cos \beta}, \qquad c = \frac{k^2 + \sin \beta}{k + k \sin \beta - k_1 \cos \beta}$$

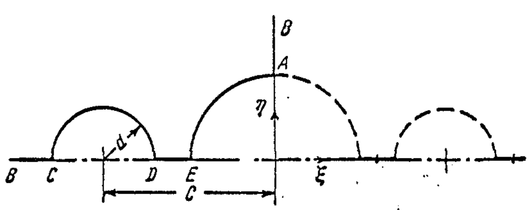

Fig. 3.

In order to map this region onto the upper half-plane we continue it by the symmetry principle into the first quadrant. The function which realizes the conformal mapping onto the upper half-plane will be an automorphic function whose analytic image has the form [5, 6]

$$\omega = \frac{\zeta}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha_n z + \beta_n}{\gamma_n z + \delta_n} - \frac{\alpha_n}{\beta_n} \right) \tag{3.1}$$

Consequently, ω is an automorphic function whose group is formed by bilinear substitutions. These substitutions are determined from various products of the basic substitutions

$$\begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} c & d^2 - c^2 \\ 1 & -c \end{Vmatrix}, \quad \begin{Vmatrix} -c & d^2 - c^2 \\ 1 & c \end{Vmatrix}$$

The general form of the mapping function (3.1) can be represented also as

$$\omega=\frac{1}{2}\left(\zeta+\frac{1}{\zeta}\right)+\frac{d^2\zeta}{\zeta^2-c^2}+\sum_{n=0}^{\infty}\frac{\mu_n\zeta}{\zeta^2-\xi_n^2}$$

$$\xi_n=\frac{\delta_n}{\gamma_n},\quad \mu_n=\frac{\beta_n\gamma_n-\alpha_n\delta_n}{\gamma_n^2}\qquad (n=0,1,2,\ldots) \tag{3.2}$$

We find the coefficients μ_n and ξ_n, calculating a series of various products of the basic substitutions

$$\xi_0=\frac{1}{c},\quad \xi_{2n}=\frac{1}{\xi_{2n-1}},\quad \mu_0=-\frac{d^2}{c^2},\quad \mu_{2n}=-\frac{\mu_{2n-1}}{\xi_{2n-1}^2}\quad (n=1,2,\ldots)$$

$$\xi_1=c-\frac{d^2}{c},\qquad \xi_3=c-\frac{d^2}{2c},\qquad \xi_{5,7}=c-\frac{d^2}{c\pm\xi_0}$$

$$\xi_9=c-\frac{d^2}{c+\xi_1},\quad \xi_{11,13}=c-\frac{d^2}{c\pm\xi_2},\quad \xi_{15}=c-\frac{d^2}{c+\xi_3}$$

$$\xi_{17,19}=c-\frac{d^2}{c\pm\xi_4},\quad \xi_{21}=c-\frac{d^2}{c+\xi_5},\quad \xi_{23,25}=c-\frac{d^2}{c\pm\xi_6}$$

$$\xi_{27}=c-\frac{d^2}{c+\xi_7},\quad \xi_{29,31}=c-\frac{d^2}{c\pm\xi_8},\quad \xi_{33}=c-\frac{d^2}{c+\xi_9}$$

$$\mu_1=-\frac{d^2}{c^2},\qquad \mu_3=-\frac{d^4}{4c^2},\qquad \frac{\mu_{5,7}}{\mu_0}=\frac{-d^2}{(c+\xi_1)^2}$$ /635

$$\frac{\mu_9}{\mu_1}=\frac{-d^2}{(c+\xi_1)^2},\quad \frac{\mu_{11,13}}{\mu_2}=\frac{-d^2}{(c\pm\xi_2)^2},\quad \frac{\mu_{15}}{\mu_3}=\frac{-d^2}{(c+\xi_3)^2}$$

$$\frac{\mu_{17,19}}{\mu_4}=\frac{-d^2}{(c\pm\xi_4)^2},\quad \frac{\mu_{21}}{\mu_5}=\frac{-d^2}{(c+\xi_5)^2},\quad \frac{\mu_{23,25}}{\mu_6}=\frac{-d^2}{(c\pm\xi_6)^2}$$

$$\frac{\mu_{27}}{\mu_7}=\frac{-d^2}{(c+\xi_7)^2},\quad \frac{\mu_{29,31}}{\mu_8}=\frac{-d^2}{(c\pm\xi_8)^2},\quad \frac{\mu_{33}}{\mu_9}=\frac{-d^2}{(c+\xi_9)^2}$$

etc.; the law of formation of the coefficients is obvious. The series converges faster the smaller the valve of the parameter d/c contained in (3.2).

Given here is a simultaneous solution of the problem of flow past three cylinders of radii d, 1 and d. The function $\omega(\xi)$ is the complex velocity potential, μ_n is the power of the doublet, and ξ_n is the coordinate into which it is imbedded. The intensity of the images of the doublet decreases; in fact, it decreases very rapidly with decreasing d/c. If in the expressions for μ_n the squares are replaced by cubes, we get the potential of the flow past the three spheres. A partial case of this problem, namely, flow past two spheres, has already been solved by Stokes by the method of successive approximation, in which doublets of given power μ_n were placed at the points of inversion ξ_n relative to the two spheres [7].

The final expression for the function which maps the original curvilinear pentagon onto the upper half-plane has the form

$$w=f(z)=-\omega^2\left(\frac{1-(k+ik_1)z}{z-k-ik_1}\right)$$

4. Formulation and Solution of the Hilbert Problem. We introduce the function

$$P(w) = \frac{\partial p}{\partial \sigma} + i\frac{\partial p}{\partial \tau}$$

which is regular in the upper half-plane $w = \tau + i\sigma$ and satisfies the condition

$$a(\tau)\frac{\partial p}{\partial \sigma} + b(\tau)\frac{\partial p}{\partial \tau} = 0.$$

on the real axis. Here

$$\begin{array}{lll} a = 0, & b = 1, & -\infty < \tau < -(c+d)^2 \\ a = \vartheta(\theta, \theta_0), & b = 1, & -(c+d)^2 < \tau < -(c-d)^2 \\ a = 0, & b = 1, & -(c-d)^2 < \tau < -1 \\ a = 1, & b = 0, & -1 < \tau < 0 \\ a = \vartheta(\theta, 0), & b = 1, & 0 < \tau < \infty \end{array}$$

where

$$\operatorname{tg}\theta = \frac{\operatorname{Im} f^{-1}(\tau)}{\operatorname{Re} f^{-1}(\tau)} = \lambda(\tau), \qquad z = f^{-1}(w)$$

The coefficients a and b have discontinuities of the first kind at the points $\tau = -1$ and $\tau = 0$, and the coefficient a also has discontinuities of the second kind at the points $\tau_1 \in (0, \infty)$, τ_2 and $\tau_3 \in [-(c+d)^2, -(c-d)^2]$; the points τ_1, τ_2 and τ_3 corresponds to the points N, L and Q of the plane z.

The exchange of function /636

$$P(w) = \frac{1}{\sqrt{w(w+1)}} P_1(w)$$

where $\sqrt{w(w+1)}$ is any branch regular in the plane with discontinuity on the real axis, eliminates the discontinuities of the first kind at the points $\tau = -1$ and $\tau = 0$ (see, e.g., [4]). The Hilbert problem is solved by reduction to the Riemann problem and the points τ_1, τ_2 and τ_3 will be singular points [8, 9]. Then the index of the problem $\varkappa = 3$ and the solution of the Hilbert problem having a second-order zero at infinity has the form

$$P(w) = \frac{c_0 + c_1 w + c_2 w^2}{(w+i)^3 \sqrt{w(w+1)}} \exp\frac{1}{2\pi i}\int_{-\infty}^{\infty} \ln\left[\left(\frac{s+i}{s-i}\right)^{\varkappa} G(s)\right]\frac{ds}{s-w}$$

Here

$$
\begin{array}{ll}
G(\tau) = 1, & -\infty < \tau < -(c+d)^2 \\
G(\tau) = \Theta(\theta, \theta_0), & -(c+d)^2 < \tau < -(c-d)^2 \\
G(\tau) = 1, & -(c-d)^2 < \tau < 0 \\
G(\tau) = \Theta(\theta, 0), & 0 < \tau < \infty
\end{array}
$$

$$
\Theta(\theta, 0) = \frac{1 - i\vartheta(\theta, 0)}{1 + i\vartheta(\theta, 0)}, \qquad \Theta(\theta, \theta_0) = \frac{1 - i\vartheta(0, \theta_0)}{1 + i\vartheta(\theta, \theta_0)}, \qquad \operatorname{tg}\theta = \lambda(\tau)
$$

$$
\ln\left(\frac{\tau + i}{\tau - i}\right)^{\varkappa} G(\tau) = \varkappa \ln\frac{\tau + i}{\tau - i} + \ln G(\tau)
$$

where by $\ln(\tau + i)(\tau - i)^{-1}$ is meant a branch which varies continuously on the real axis (including the point at infinity), except for a certain point $\tau_0 \in (-\infty, \infty)$ which does not coincide with any of the points of discontinuity of the coefficients a and b, and $\ln G(\tau)$ is determined according to the following rule:

$$
\arg\frac{G(\tau_n - 0)}{G(\tau_n + 0)} = 0 \qquad (n = 1, 2, 3)
$$

The real constants c_0, c_1 and c_2 are determined from conditions (2.1) and (2.2).

The solution in the plane z has the form

$$
\frac{\partial p}{\partial \nu} + i\frac{\partial p}{\partial \mu} = \left(\frac{\partial p}{\partial \sigma} + i\frac{\partial p}{\partial \tau}\right) f'(z) = \frac{c_0 + c_1 f(z) + c_2 f^2(z)}{(f(z) + i)^3 \sqrt{f(z)(f(z) + 1)}} \times
$$

$$
\times f'(z)\left(\exp\frac{3}{2\pi i}\int_{\Gamma} \ln\frac{f(s) + i}{f(s) - i}\,\frac{f'(s)\,ds}{f(s) - f(z)} + \right.
$$

$$
\left. + \exp\frac{1}{2\pi i}\int_{AB} \ln\Theta(\theta, 0)\frac{f'(s)\,ds}{f(s) - f(z)} + \exp\frac{1}{2\pi i}\int_{CD} \ln\Theta(\theta, \theta_0)\frac{f'(s)\,ds}{f(s) - f(z)}\right)
$$

Γ — is the contour of ABCDE

on AB $(0 < \theta < \theta_1,\ \theta_1 = \arccos k)$

$$
r\exp i\theta = k^{-1}(\cos\theta - \sqrt{\cos^2\theta - k^2})\exp i\theta
$$

on CD $(\theta_0 - \theta_1 < \theta < \theta_0 + \theta_1)$

$$
r\exp i\theta = k^{-1}(\cos(\theta - \theta_0) - \sqrt{\cos^2(\theta - \theta_0) - k^2})\exp i\theta
$$

The pressure is computed by the formula /637

$$
p = \operatorname{Im}\int_{-1}^{z}\left(\frac{\partial p}{\partial \nu} + i\frac{\partial p}{\partial \mu}\right) dz + p_2 = \operatorname{Im}\int_{-1}^{w}\left(\frac{\partial p}{\partial \sigma} + i\frac{\partial p}{\partial \tau}\right) dw + p_2
$$

Having determined the pressure, all the remaining unknown functions can also be found in closed form. For example, the shape of the diffracted shock wave AB is computed from (1.1) by the formula

$$\psi(y) = \frac{(\gamma+1)(y-k_1)}{4kk_1} E_1 + \frac{\psi(k_1)}{k_1} y - \frac{\gamma+1}{4k} y \int_{k_1}^{y} s^{-2} p(s)\, ds$$

$\psi(k_1)$ being known from the solution of the problem in region 4.

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